THE PRINCIPLE OF THE MINIMUM ACTION IN TRUE TIME

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ABSTRACT*:*
THE PRINCIPLE OF THE MINIMUM ACTION IS ONE OF THE MOST GENERAL LAWS OF NATURE (HAMILTON W., 1834), WHICH CLAIMS THAT FROM ALL THE POSSIBLE WAYS A SYSTEM COULD FLUCTUATE FROM ONE CONFIGURATION TO ANOTHER, IN PRACTICE IT IS REALIZED THE ONE FOR WHICH THE ACTION IS MINIMUM. IN THIS WORK WE SHALL WRITE THE LAGRANGEAN (LAGRANGE'S OPERATOR) FOR A RELATIVISTIC PARTICLE AND FROM THIS THE RELATIVIST PHYSIC VALUES WILL BE DEDUCED.

1. The principle of minimum action

The fundamental observation that underlies the introduction of the notion of action, is that Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right)-\frac{\partial L}{\partial q_k}=Q_k\ ,\qquad k=1,2,\ldots.,h \qquad (1)$$

where $L=L(q_k,\dot{q}_k,t)$ is the Lagrange function in coordinates q_k and the velocities $\dot{q}_k$ generalized,
is a necessary condition for the function:

$$S=\int_{t_1}^{t_2} L(q_k,\dot{q}_k,t)dt \qquad (2)$$

to be minimal, which will be demonstrated in the following theorem.

The value S is precisely the action of the system.

The problem of minimizing the function (2) is a specific problem for the variational calculation (see [2]) and consists in finding the functions $q_k:[t_1,t_2]\to\mathbb{R}\,, k=1,2,\ldots,h$, where S has the minimum value.

Because equations are established for olonome systems, we will analyze the principle of minimum action only for such systems. In addition, because the conditions necessary to minimize the action, known as Euler's equations, have the form (1), with $Q_k=0,\ k=1,2,\ldots,h$, the principle of minimum action can only be applied in the absence of non-conservative forces.

Theorem 1. Considering $\mathcal{X}$ a part of the space $\underset{k=1}{\overset{h}{\times}} C^2_{\mathbb{R}}([t_1,t_2])$ defined by the following conditions $q_k(t_1) = a_k$, $q_k(t_2) = b_k$, for all $k = 1,2,\dots,h$.
Then, considering $S: \mathcal{X} \to R$ as a function like:

$$S(q_1,q_2,\dots,q_h) = \int_{t_1}^{t_2} L(q_1,q_2,\dots,q_h,\dot{q}_1,\dot{q}_2,\dots,\dot{q}_h;t)dt \qquad (3)$$

where $L \in C^2_{\mathbb{R}}$. In this condition, so that the element $(\bar{q}_1,\bar{q}_2,\dots,\bar{q}_h) \in \mathcal{X}$ be point of extreme for S, it is necessary that it verifies Euler's equation

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0\,, \quad k = 1,2,\dots,h. \qquad (4)$$

Demonstration. Supposing that the element $(\bar{q}_1,\bar{q}_2,\dots,\bar{q}_h)$ is the extreme for S and considering some increases of this function, under the form $\delta_1(t) = \alpha_1\eta_1(t),\dots,\delta_h(t) = \alpha_h\eta_h(t)$ where $\alpha_1,\dots,\alpha_h \in \mathbb{R}$ and $\eta_1,\dots,\eta_h \in C^2_{\mathbb{R}}$, that $\eta_k(t_1) = \eta_k(t_2)$ for every $k = 1,2,\dots,h$. Then the function $I : \mathbb{R}^h \to \mathbb{R}$, whose values are expressed by the formula

$$I(\alpha_1,\dots,\alpha_h) = \int_{t_1}^{t_2} L(\bar{q}_1 + \alpha_1\eta_1,\dots,\bar{q}_h + \alpha_h\eta_h;\dot{\bar{q}}_1 + \alpha_1\dot{\eta}_1,\dots,\dot{\bar{q}}_h + \alpha_h\dot{\eta}_h;t)dt \qquad (5)$$

will have an extreme in the point $\alpha_1 = \alpha_2 = \cdots = \alpha_h = 0$, so in this point we will have

$$\frac{\partial I}{\partial \alpha_1} = \frac{\partial I}{\partial \alpha_2} = \cdots = \frac{\partial I}{\partial \alpha_h} = 0\,. \qquad (6)$$

Explaining these derivatives we obtain for $\bar{q}_1,\bar{q}_2,\dots,\bar{q}_h$ the following conditions:

$$\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_k}\eta_k + \frac{\partial L}{\partial \dot{q}_k}\dot{\eta}_k\right)dt = 0\ , \qquad k = 1,2,\dots,h\,.$$

Calculating the integral of the second term using integration formula through parts and taking into account the fact that the functions η_k cancel each other in the points t_1 and t_2, we obtain the following conditions

$$\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\right)\eta_k dt = 0\ , \quad k = 1,2,\dots,h\,. \qquad (7)$$

and because of the fact that here the functions η_k are arbitrary, the integral can be zero only if the point $(\bar{q}_1,\bar{q}_2,\dots,\bar{q}_h)$ the equations (4) can be verified. ■

2. Lagrangean of the relativistic particle

The Lagrangean of the relativistic particle can now be deduced from the conditions that naturally impose the action of the system formed only by this particle.

Since the trajectory of the free particle in Minkowski space is rectilinear, a function that is extreme on this trajectory is the length of the arcs measured using the temporal metric.

It follows that the expression of the action must be

$$S = -\alpha \int_{e_1}^{e_2} d\tau \ , \qquad (8)$$

where $\alpha > 0$, e_1 and e_2 are events (in a causal relationship) between which the particle evolves, and τ is its own time. The minus sign appears if we want the action extreme to be minimal.

Replacing the expression of our own time, we find:

$$S = -\alpha c \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt \ , \qquad (9)$$

then the lagrangean searched must be of the form:

$$L = -\alpha c \sqrt{1 - \frac{v^2}{c^2}} + K \qquad (10)$$

where K is a constant that will be determined conveniently.

From the principle of correspondence, for $v \ll c$, the lagrangean found to be reduced to that of the free particle of classical mechanics, $L = \frac{1}{2} m_0 v^2$.

By developing the relativistic lagrangean expression, we find a series of forms

$$L = -\alpha c + K + \frac{v^2}{2c} + \cdots \qquad (11)$$

so that, to comply with the principle of correspondence, is required $K = \alpha c$ and $\alpha = m_0 c$. With these values, the relativist lagrangean becomes

$$L = (m - m_0)c^2 \sqrt{1 - \frac{v^2}{c^2}} \ . \qquad (12)$$

The constant $K = m_0 c^2$, that appears as a reference energy when establishing correspondence with classical mechanics is usually neglected because it does not affect either the principle of minimum action or the expressions of physic values generalized impulse and generalized force, which are obtained by derivations. In conclusion, the lagrangean of the free relativistic particle has the form

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \ , \qquad (13)$$

and its action is:

$$S = -m_0 c^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt \ . \qquad (14)$$

From these results we can deduce the components of the impulse which are

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{x}_k} = \frac{m_0 \dot{x}_k}{\sqrt{1 - \frac{v^2}{c^2}}} \ , \qquad k = 1,2,3 \ , \qquad (15)$$

because $v = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$. Restraining in vectorial writing, we find the following expression for impulse:

$$\bar{p} = \frac{L}{\dot{\bar{v}}} = m_0(1 - v^2 c^{-2})^{-1/2}\bar{v} \quad , \qquad (16)$$

which agrees with the expression $\bar{p} = m\bar{v}$ of the relativistic impulse, if we take into account the expression of the relativistic mass.

For total generalized energy, lagrangean given by the formula (13) leads us, according to (16), to the expression:

$$E = \bar{p}\bar{v} - L = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \,, \qquad (17)$$

which totally agrees with Einstein's formula.

Bibliografy

[1] Bălan T., Chiriac I., *Relativity in true time*, Editura Universitaria Craiova, 2001.
[2] Savelyov I., *Fundamentals of theorethical physics*, Editura Mir, Moskow, 1982