METRICS AND NORMS IN MINKOWSKI SPACE

Novac-Claudiu CHIRIAC
University "Constantin Brâncuși" of Tg-Jiu

ABSTRACT.
THE CONSTRUCTION OF EUCLIDEAN NORM AND METRICS CAN NOT BE APPLIED EXACTLY IN THE CASE OF MINKOWSKI SPACE. ON THE ONE HAND THE EXPRESSIONS BY WHICH WE DEFINE THE NORMS AND METRICS OF MINKOWSKI SPACE MUST TAKE INTO ACCOUNT THE TYPE OF THE RESPECTIVE VECTORS AND ON THE OTHER HAND FUNDAMENTAL INEQUALITIES MUST BE FORMULATED ACCORDING TO THE NATURE OF THE SIDES OF THE TRIANGLE. WE WILL MEET THESE REQUIREMENTS CONSIDERING THE NORMS AND METRICS RESTRICTED, THAT IS DEFINED ONLY FOR CERTAIN VECTORS OF SPACE.

1. Binary relations in Minkowski space

By binary relation on E we understand every part $\boldsymbol{R}$ of $E \times E$. In particular, equality on E is noted as $\Delta = \{(e,e) : e \in E\}$ and is called diagonal of $E \times E$.

The section of $\boldsymbol{R}$ in e is noted with

$\boldsymbol{R}[e] = \{f \in E : (e,f) \in \boldsymbol{R}\}$

and is also called cone with tip in e.

Definition. *We consider $(E, (\cdot,\cdot))$ a Minkowski space and Q the quadratic form attached to it. we say about the events $e_1, e_2 \in E$ that they are in a* <u>*causal relation*</u> *(or* <u>*temporal relation*</u>*), more precisely e_1 is the* <u>*cause*</u> *for e_2, or e_2 is the* <u>*effect*</u> *for e_1, if $e_1 = e_2$, or $t_1 < t_2$ and $Q(e_2 - e_1) > 0$. Noting the causal relation with $\boldsymbol{K} \subset E \times E$, we have*

$$\boldsymbol{K} = \left\{(e_1, e_2) : e_1 = e_2 \; or \, c(t_2 - t_1) > \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right\}$$

We call the signal relation $\boldsymbol{X} \subset E \times E$ defined by

$$\boldsymbol{X} = \left\{(e_1, e_2) : c(t_2 - t_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right\}$$

The relation $\boldsymbol{S} \subset E \times E$ defined by

$$\boldsymbol{S} = \left\{(e_1, e_2) : e_1 = e_2 \; or \, c|t_2 - t_1| < \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right\}$$

will be called a <u>*space relation*</u>*, or relation of* <u>*causal independence*</u>

Poposition. *Binary relations $\boldsymbol{K}$, $\boldsymbol{X}$ and $\boldsymbol{S}$ on Minkowski space E have the following properties:*

1. *$\boldsymbol{K}$ is a partial order relation compatible with the linear space structure of E*
2. *$\boldsymbol{X}$ is only antisymmetric and reflexive*
3. *$\boldsymbol{S}$ is reflexive and symmetrical*

4. $\boldsymbol{K} \cup \boldsymbol{K}^{-1} \cup \boldsymbol{X} \cup \boldsymbol{X}^{-1} \cup \boldsymbol{S} = E \times E$.

The relations defined above share vectors from the Minkowski space in a few categories:

a) position vectors of events $e \in (\boldsymbol{K} \cup \boldsymbol{K}^{-1})[0]$ are called temporal vectors. They are characterized by that $Q(e) > 0$ or $e = 0$ which is why they are also called positive vectors of the interior product $(.,.)$.

b) position vectors of events $e \in (\boldsymbol{X} \cup \boldsymbol{X}^{-1})[0]$ are called signal vectors. They are characterized by that $Q(e) = 0$ and they are also called neutral vectors of the interior product $(.,.)$.

c) position vectors of events $e \in \boldsymbol{S}[0]$ are called spatial vectors. They are characterized by that $Q(e) < 0$ or $e = 0$ they are also called negative vectors of the interior product $(.,.)$.

2. Indefinite norms and metrics

If on a linear space E is defined a scalar product $\langle \cdot,\cdot \rangle : E \times E \to \Gamma$, where $\Gamma = \mathbb{R}$ or $\Gamma = \mathbb{C}$, through a standard construction is obtained a norm $\|\cdot\| : E \to \mathbb{R}$, as well as a metric $d: E \times E \to \mathbb{R}$ considering $\|x\| = \langle x, x \rangle^{1/2} = [Q(x)]^{1/2}$ and $d(x,y) = \|x - y\|$. Norm and metrics construction is possible because Q is a positive defined form, fact explained by the axioms of the scalar product. This construction is not realizable exactly in this way in the case of Minkowski spaces, where the interior product $(e,e) = Q(e)$ can be both positive and negative but also zero for $e \neq 0$. Another reason why this construction can not be practiced in Minkowski spaces is that the rule of the triangle is sometimes inverted, that is, it does not take place constantly inequality "one side is less than the sum of the other two", even if we limit ourselves to considering triangles for which side lengths are real numbers that is, the sides are temporal type. It can be noted that neither inequality "one side is great than the other two" can not be written for all sides of the triangle, because making the sum of these inequalities and simplifying with the perimeter of the triangle we get the impossible inequality $1 \geq 2$.

In conclusion, on the one hand the expressions by which we define the norms and metrics of Minkowski space must take into account the type of the respective vectors and on the other hand fundamental inequalities must be formulated according to the nature of the sides of the triangle. We will meet these requirements considering the norms and metrics restricted, that is defined only for certain vectors of space, determined by previous binary relationships. Thus the norms will not be defined throughout the space E and the metrics will not be defined throughout the space $E \times E$, but they will be restricted to parts of these spaces, which are naturally determined by their binary relations.

Definition. *We call temporal indefinite norm on Minkowski space E functional* $\|\cdot\|_t : \boldsymbol{K}[\boldsymbol{0}] \to \mathbb{R}_+$, *defined by expression:*

$$\|e\|_t = (e,e)^{1/2} = \sqrt{c^2t^2 - x^2 - y^2 - z^2} \quad (1)$$

We call spatial indefinite norm on Minkowski space E functional $\|\cdot\|_s : \boldsymbol{S}[\boldsymbol{0}] \to \mathbb{R}_+$, *defined by expression:*

$$\|e\|_s = [-(e,e)]^{1/2} = \sqrt{x^2 - y^2 - z^2 - c^2t^2} \quad (2)$$

Functional $k : \boldsymbol{K} \to \mathbb{R}_+$ *, expressed by formula:*

$$k(e_1, e_2) = \|e_2 - e_1\|_t \quad (3)$$

is called temporal indefinite metric and functional $\xi : \boldsymbol{S} \to \mathbb{R}_+$ *, expressed by formula:*

$$\xi(e_1, e_2) = \|e_2 - e_1\|_s \quad (4)$$

is called spatial indefinite metric.

Terminology is determined by the classification of event position vectors.

Next we will agree that with writing $\|e\|_t$, $k(e_1, e_2)$, etc. , let's suppose that $e \in \boldsymbol{K}[\boldsymbol{0}]$, $(e_1, e_2) \in \boldsymbol{K}$, etc., such that these symbols make sense.

Proposition. *The indefinite norms of any Minkowski space have the following properties:*

$[T_1]$ $\|e\|_t = 0$ *if and only if* $e = 0$ *;*

$[T_2]$ $\|\lambda e\|_t = \lambda \|e\|_t$ *for any* $\lambda \geq 0$ *;*

$[T_3]$ $\|e_1 + e_2\|_t \geq \|e_1\|_t + \|e_2\|_t$ *with equality if and only if* e_1 *and* e_2 *have the position vectors collinear ;*

$[S_1]$ $\|e\|_s = 0$ *if and only if* $e = 0$ *;*

$[S_2]$ $\|\lambda e\|_s = |\lambda| \|e\|_s$ *for any* $\lambda \in \mathbb{R}$.

Demonstration. Properties $[T_1]$ and $[S_1]$ is essentially based on the definition of relations $\boldsymbol{K}$ and $\boldsymbol{S}$. Properties $[T_2]$ and $[S_2]$ results from the expressions (1) and (2), mentioning that $\lambda \geq 0$ in $[T_2]$ is required because the section $\boldsymbol{K}[\boldsymbol{0}]$ is a sharp con. To demonstrate $[T_3]$ we notice that, according to the formula (3), we have ($e_1 = 0$ or $(e_1, e_1) > 0$) and ($e_2 = 0$ or $(e_2, e_2) > 0$). In truth, of the imposed conditions, namely $ct_1 > \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $ct_2 > \sqrt{x_2^2 + y_2^2 + z_2^2}$, by summing result

$c(t_1 + t_2) > \sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} \geq \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2}$,

according to the triangle regulation for Euclidian metric. Otherwise we have $\boldsymbol{K}[\boldsymbol{0}] + \boldsymbol{K}[\boldsymbol{0}] \subseteq \boldsymbol{K}[\boldsymbol{0}]$ which allows us to write $[T_3]$ without the express mention of the fact that $e_1 + e_2 \in \boldsymbol{K}[\boldsymbol{0}]$.

The cases $e_1 = 0$ respectively $e_2 = 0$ lead to direct verification of the relationship $[T_3]$ even with equality. We still have to consider $(e_1, e_1) > 0$, $(e_2, e_2) > 0$ when we have and $(e_1 + e_2, e_1 + e_2) > 0$. In this case, if we note $e_1 = (t_1, x_1, y_1, z_1)$ and $e_2 = (t_2, x_2, y_2, z_2)$, we do not have $t_1 = 0$ and we do not have $t_2 = 0$. we now consider the trinomial:

$T(\lambda) = (e_1 + \lambda e_2, e_1 + \lambda e_2) = (e_1, e_1) + 2\lambda(e_1, e_2) + \lambda^2 (e_2, e_2)$.

Since $(e_2, e_2) > 0$, there will be values for λ (great) that $T(\lambda) > 0$. But writing the trinom in the form

$T(\lambda) = c^2 (t_1 + \lambda t_2)^2 - (x_1 + \lambda x_2)^2 - (y_1 + \lambda y_2)^2 - (z_1 + \lambda z_2)^2$,

it is seen that for $\lambda_0 = -t_1/t_2$ we have $T(\lambda_0) < 0$.

The trinom T will therefore have real solutions, such as

$\Delta = (e_1, e_2)^2 - (e_1, e_1)(e_2, e_2) \geq 0$. (5)

To observe now that and $(e_1, e_2) > 0$.In truth, since $e_1, e_2 \in \boldsymbol{K}[\boldsymbol{0}] \setminus \{0\}$ conclude $ct_1 > \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $t_2 > \sqrt{x_2^2 + y_2^2 + z_2^2}$, then, by multiplying and using Cauchy-Buniakovski-Schwarz inequality for the Euclidian scalar product, obtain

$c^2 t_1 t_2 > \sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2} \geq x_1 x_2 + y_1 y_2 + z_1 z_2$.

Extracting the radical in the relation (5) written in form $(e_1, e_2)^2 \geq (e_1, e_1)(e_2, e_2)$, we obtain inequality

$(e_1, e_2) \geq \|e_1\|_t \|e_2\|_t$. (6)

Proceed as usual in real spaces with scalar product, well amplifying with 2 and sum in both members expression $(e_1, e_1) + (e_2, e_2) = \|e_1\|_t^2 + \|e_2\|_t^2$, from (6) is obtained

$(e_1 + e_2, e_1 + e_2) \geq (\|e_1\|_t + \|e_2\|_t)^2$,

such as so that $[T_3]$ results from a radical extraction.

Equality in $[T_3]$ is equivalent to the null to Δ in expression (5), then with the linear dependence discussed above. Really, if $e_2 = \lambda e_1$, you will have to $\lambda \geq 0$, for do not get out of the cone $\boldsymbol{K}[\mathbf{0}]$. Calculating the norms what interfere in $[T_3]$ find
$\|e_1 + \lambda e_1\|_t = (1+\lambda)\|e_1\|_t = \|e_1\|_t + \|\lambda e_1\|_t$
consequently condition $[T_3]$ is verified having equality.

Reciprocally, if $\|e_1 + e_2\|_t = \|e_1\|_t + \|e_2\|_t$, we deduce
$\Delta = (e_1, e_2)^2 - (e_1, e_1)(e_2, e_2) = 0$. Substituting $(e_1, e_2) = \|e_1\|_t \|e_2\|_t$, obtain
$T(\lambda) = (\|e_1\|_t + \lambda\|e_2\|_t)^2$ which shows that the trinom T can not be strictly negative. Now returning to the expression
$T(\lambda) = c^2(t_1 + \lambda t_2)^2 - (x_1 + \lambda x_2)^2 - (y_1 + \lambda y_2)^2 - (z_1 + \lambda z_2)^2$,
it is noted that, if would not null and the other brackets, for $\lambda_0 = -t_1/t_2$, which null the first one, we would have $T(\lambda_0) < 0$. So it follows $x_1 + \lambda x_2 = 0$, $y_1 + \lambda y_2 = 0$, $z_1 + \lambda z_2 = 0$ therefore $e_1 + \lambda e_2 = 0$. ■

Consequence. *Indefinite metrics k and ξ of a Minkowski space have the following properties:*
(MT_1) $k(e_1, e_2) = 0$ *if and only if* $e_1 = e_2$ *;*
(MT_2) $k(e_1, e_3) \geq k(e_1, e_2) + k(e_2, e_3)$ *;*
(MS_1) $\xi(e_1, e_2) = 0$ *if and only if* $e_1 = e_2$ *;*
(MS_2) $\xi(e_1, e_2) = \xi(e_2, e_1)$.

Demonstration. The demonstration follows the techniques used to deduce the properties of usual metrics in metric spaces ■

Now we will specify what it represents in practice values of indefinite metrics calculated with formulas (3) and (4), namely:

1. Values of temporal metric k represents times. More precisely $k(e_1, e_2)$ is the time it registers an inertial observer which evolves between these two events (what are they in relation of the cause). depending on the observer, this time is also called its proper time, that is proper inertial observer respectively, that realizes evolution. In fine k measure time.

2. Values of spatial metric ξ represents distances. For events e_1 and e_2 simultaneous for an inertial observer, $\xi(e_1, e_2)$ is the distance between the places where these events are produced, therefore ξ measures space.

Bibliografy

[1] Callahan J., *The Geometry of Spacetime*, Springer-Verlag, 2000.
[2] Kronheimer E. H., Penrose R., *On the structure of causal Spaces*, Prog. Camb. Phil. Soc. **63** (1967), p. 481-501.
[3] Naber G., *The Geometry of Minkowski Spacetime*, Springer-Verlag, 1992.
[4] Nanda S., *Topology for Minkowski spaces*, J. Math. Phys. Vol. **12**, No. 3(1971), p. 394-401.
[5] Zeeman E.C., The topology of Minkowski space, Topology, vol. **6** (1967), p. 161-170.