# HORISTOLOGICAL vs TOPOLOGICAL DISCRETENESS

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**ABSTRACT**
*We survey the problem of discreteness from three points of view: that of structures, functions and sets. Then we compare several results on discreteness in topological and horistological framework to conclude that the horistologies are better suited for the study of discreteness.*

## 1. Structures of continuum and discreteness

**1.1. The general notion of continuity**. It is widely accepted that the notion of continuity requires a remarkable effort of abstraction, which usually involves infinity. However, as a matter of fact we refer to continuity in every domain of activity, from our everyday practice to art and philosophy. Among these fields, mathematics distinguishes by offering a structural base to the study of continuity, which is represented by the topological structures. In other terms, whenever we speak of continuity in mathematical sense, there is some topological structure in the background, even if it is sometimes deliberately ignored. By extension to the natural sciences, which allow sufficient mathematization, the criterion of establishing whether a phenomenon is continuous or not is based on measuring the efficiency of the topological tools in its study. The topological structures give the possibility of developing a qualitative study, which corresponds to the most general idea of continuity.

For the sake of an exact comparison with horistology, we recall the most intuitive definition of a topology (see [BN], [PG], etc.):

**1.2. Definition**. Let $S$ be a non-void set. Function $\tau: S \longrightarrow \mathcal{P}(\mathcal{P}(S))$ is called *topology* on $S$ if it satisfies the conditions

[TOP$_1$] $x \in V$ for all $V \in \tau(x)$

[TOP$_2$] If $V \in \tau(x)$ and $\supseteq V$ , then $U \in \tau(x)$

[TOP$_3$] If $, V \in \tau(x)$ , then $U \cap V \in \tau(x)$

[TOP$_4$] $(\forall)\ V \in \tau(x),\ (\exists)\ W \in \tau(x)$ such that $V \in \tau(x)$ for all $y \in W$.

The elements of $\tau(x)$ are called *neighborhoods* of $x$, and the pair $(S, \tau)$ is named *topological space*. In other words, we define a topology on $S$ by specifying the family of neighborhoods at each point of $S$. Conditions [$TOP_2$] and [$TOP_3$] show that each family $\tau(x)$ of neighborhoods forms a filter in $\tau(x)$.

We mention that besides $\tau$, there are many equivalent ways of defining the topology. The above definition describes a "standard" topology, but in the specialized literature we may find plenty of variants and extensions.

The subsequent example of topology makes use of finite sets:

**1.3. Example**. Let $S$ be an arbitrary non-void (and essentially infinite) set. Function $\tau: S \longrightarrow \mathcal{P}\big(\mathcal{P}(S)\big)$, defined by.

$\tau(x) = \{V \in \mathcal{P}(S): x \in V \; and \; card \; \complement V \in \mathbb{N}\}$.

is a topology on $S$, known as *topology of finite complements*.

**1.4. General features of discreteness**. Most frequently, especially in practical problems and computing, discreteness reduces to finiteness. An old example concerns the division of a body into parts: if we accept the existence of some elementary (i.e. indivisible) particles, then the process shall find its end in a finite number of steps, hence the number of parts will be finite. Nowadays, the computers furnish another remarkable example of discreteness, which also reduces to finiteness. This customary identification of the two notions misleads to the illusion that discreteness does not need a structural framework. However, discreteness and infiniteness may naturally coexist, especially in mathematics.

At a structural level, it is impossible to describe discreteness as opposite to (or negation of) topological structures, because the opposite of "continuous" is "discontinuous", and both make sense inside the same structure. The only place where topological structures meet discreteness is the particular case of so-called *discrete topology*, in which membership $x \in V$ is enough to qualify $V$ as a neighborhood of $x$. On the other hand, the topological structures allow dual structures (introduced in [$BT_2$] in relativist terms of events, worlds, causality, etc. and called *horistologies*), namely:

**1.5. Definition**. Let $W$ be a non-void set. Function $\chi: W \longrightarrow \mathcal{P}\big(\mathcal{P}(W)\big)$ is called *horistology* on $W$ if it satisfies the conditions:

[$HOR_1$] $e \notin P$ for all $P \in \chi(e)$

[$HOR_2$] If $P \in \chi(e)$ and $Q \subseteq P$, then $Q \in \chi(e)$

[$HOR_3$] If $P, Q \in \chi(e)$, then $P \cup Q \in \chi(e)$

[$HOR_4$] $(\forall)\; P \in \chi(e), (\exists)\; H \in \chi(e)$ such that $[l \in P \; and \; Q \in \chi(l)] \Longrightarrow [Q \subseteq H]$.

The elements of $\chi(e)$ are called *perspectives* of $e$, and the pair $(W, \chi)$ is named *horistological world*. In other words, we define a horistology on $W$ by specifying the family of perspectives at each event of $W$. Similarly to topologies, we may define horistologies by specifying some families of parts, set operators, classes of nets, etc. (see [$BT_2$], [PM], [CI]).

The purpose of this paper is to present a collection of facts, which sustain the idea that the horistologies deserve the role of structures of discreteness. We start by the property that, unlike topological structures, the horistologies are always accompanied with order relations (like *causality* in relativity):

**1.6. Horistologies generate orders**. If $(W,\chi)$ is a horistological world, then
$K(\chi)=\{(e,l)\in W^2: either\ e=l\ or\ \{l\}\in\chi(e)\}$.
is an order relation on $W$. We call it *proper order* of $\chi$.
**1.7. Example**. Let $W$ be a non-void world, and let $K\subseteq W\times W$ be an order on $W$. As usually, $\delta$ stands for equality, $=K^\circ\setminus\delta$ , and
$K[x]=\{y\in W:(x,y)\in K\}$,
represents the *section* of $K$ at $x$. If we define $\chi:W\longrightarrow\mathcal{P}(\mathcal{P}(W))$ by
$\chi(e)=\{P\in\mathcal{P}(W): P\subseteq K^\circ[e]\ and\ card\ P\in\mathbb{N}\}$,
Then $\chi$ is a horistology on $W$, for which $K(\chi)=K$.

Of course, we may take here $W=\mathbb{R}$, which allows many other horistologies. The multitude of horistological structures on $\mathbb{R}$ imposes a reconsideration of its continuity. From a structural point of view, we take it, in fact, as a pattern of continuum, but only if endowed with a topology (most frequently Euclidean). Paper [P-C] contains a horistological construction of $\mathbb{R}$.

Examples of di¤erent horistologies, which have the same proper order, show that horistologies carry more information than orders.

**1.8. Correspondence through ∁.** Conditions [HOR$_2$] and [HOR$_3$] show that each family $\chi(e)$ of perspectives forms an ideal in $\mathcal{P}(W)$. Generally speaking, there is a strong connection between filters and ideals, realized by the operation ∁ of complementation (e.g. see [R-S]). More exactly, in an arbitrary set $S$, if $\mathcal{I},\mathcal{F}\subseteq P(S)$, then
$[\mathcal{I}\ is\ an\ ideal]\Leftrightarrow[\mathcal{F}=\{\complement P: P\in\mathcal{I}\}\ is\ a\ filter]$.
This relation suggests a simple way of deriving horistologies from topologies, and vice-versa. In fact, this relation "$filter\leftrightarrow ideal$" sometimes gives a reciprocal determination of a topology by a horistology, as for example in $\mathbb{R}$, where $K$ is the natural order, and is defined by
$[P\in\chi(x)]\Leftrightarrow[\exists\varepsilon>0\ such\ that\ P\subseteq(x+\varepsilon,\infty)]$.
The corresponding topology is defined by
$[V\in\tau(x)]\Leftrightarrow[\exists\varepsilon>0\ such\ that\ V\supseteq(-\infty,x+\varepsilon)]$.

We may easily extend this correspondence to totally ordered spaces, but it is not valid for all topologies, respectively horistologies. For example, if $\tau$ is the topology of finite complements in $\mathbb{R}$, then function $\chi:\ \mathbb{R}\longrightarrow\mathcal{P}(\mathcal{P}(\mathbb{R}))$, of values
$\chi(x)=\{P\in\mathcal{P}(\mathbb{R}):\ \complement P\in\tau(x)\}$,
does not satisfy [HOR$_4$]. Consequently, we conclude that the correspondence through ∁ is not generally valid.

**1.9. Uniform horistologies**. It is well known that notions like uniformly continuous functions, fundamental sequences, etc., involve neighborhoods of the same size for different points. Similarly to topological structures, there exist particular types of horistologies, where we may compare by "size" the perspectives of different events. More exactly, a uniform horistology (briefly u.h.) is a family $\mathcal{H}\subseteq\mathcal{P}(W^2)$, which satisfies the conditions:
[uh$_1$] $\pi\cap\delta=\phi$ for all $\pi\in\mathcal{H}$
[uh$_2$] If $\pi\in\mathcal{H}$ and $\lambda\subseteq\pi$, then $\lambda\in\mathcal{H}$

[uh$_3$] If $\lambda, \pi \in \mathcal{H}$, then $\lambda \cup \pi \in \mathcal{H}$
[uh$_4$] $(\forall)\ \pi\ (\exists\ \lambda)$ such that $[\omega \in \mathcal{H}] \Longrightarrow [\lambda \supseteq \pi \circ \omega\ and\ \lambda \supseteq \omega \circ \pi]$.

The elements of $\mathcal{H}$ are called *prospects*, and the pair $(W, \mathcal{H})$ represents a *uniform horistological world*.

It is easy to see that if $(W, \mathcal{H})$ is a uniform horistological world, then function $\chi_{\mathcal{H}}: W \longrightarrow \mathcal{P}(\mathcal{P}(W))$ of values

$\chi_{\mathcal{H}}(e) = \{P \in \mathcal{P}(W) :\ (\exists)\ \pi \in \mathcal{H}\ such\ that\ P \subseteq \pi(e)\}$,
is a horistology on $W$. In addition, for the proper order of $\mathcal{H}$ we have
$K(\mathcal{H}) \stackrel{\text{def}}{=} [\cup \{\lambda :\ \lambda \in \mathcal{H}\}] \cup \delta = K(\lambda_{\mathcal{H}})$.

Examples like 1.7. show that not all horistologies are uniformizable. We may endow $\mathbb{R}$ with simple u. horistologies, e.g.
$\mathcal{H} = \{\pi \in \mathcal{P}(\mathbb{R}^2) :\ (\exists)\ \varepsilon > 0\ such\ that\ [(x, y) \in \pi] \Longrightarrow [y - x > \varepsilon]\}$.
Similarly to topological structures, the horistologies and the uniform horistologies represent qualitative structures. At a quantitative level, which is specific to practical measurements, the duality topology / horistology corresponds to sub-additive / super-additive metrics and norms.

**1.10. Super-additive metrics**. An important type of $(u.)$ horistologies is generated by super-additive (briefly S.a.) metrics. If $\Pi \subset W \times W$ is an order on $W$, then $\rho : \Pi \longrightarrow \mathbb{R}_+$ is a *S.a. metric* if it satisfies the conditions:
[M1] $\rho(e_1, e_2) = 0 \Leftrightarrow e_1 = e_2$
[M2] $\rho(e_1, e_3) \geq \rho(e_1, e_2) + \rho(e_2, e_3)$ for all $(e_1, e_2), (e_2, e_3) \in \Pi$ .

We say that $\rho$ is a *S.a. pseudo metric* if in [M1] only " $\Longleftarrow$" holds. The triplet $(W, \Pi, \rho)$ is called *S.a*. (respectively *pseudo*) *metric world*.

Because of [M2], we cannot define S.a. metrics on the whole $\times W$ . Further restrictions to $\Lambda \subset \Pi$ are always possible, but the converse process is more significant. A variant of such prolongations is studied in [BT$_1$].

An interesting example of a S.a. metric appears in Propositional Calculus. In the set $W$ of all propositions with $n$ elementary sentences, the relation of logical implication, $A \Longrightarrow B$, here noted $(A, B) \in \Pi$ , means that proposition $B$ is true whenever $A$ is. Function , which counts the "intermediate" propositions, is a S.a. metric. More explicitly, if $\alpha$ and $\beta$ are the numbers of cases when $A$, respectively $B$ are true (in the corresponding "truth tables"), then

$$\rho(A, B) = 2^{\beta-\alpha-1}.$$

Each (pseudo) metric world naturally carries a uniform horistology $\mathcal{H}_\rho$ , which is generated by hyperbolic prospects of various radii $r > 0$,

$$\pi_r = \{(e_1, e_2) \in \Pi :\ \rho(e_1, e_2) > r\}.$$

More exactly, $\pi \in \mathcal{H}_\rho$ if and only if $\pi \subseteq \pi_r$ for some $r > 0$.

Theorem 9 in [BT$_2$] gives a characterization of the metrizable u. horistologies in terms of exhausting prospects.

**1.11. Super-additive norms**. In real linear spaces, we may express the above structural duality in terms of sub / super - additive norms. If $\Pi \subset W \times W$ is an order on $W$, then $|\cdot| : \Pi[0] \longrightarrow \mathbb{R}_+$ is a *super-additive* (briefly *S.a.*) *norm* if it satisfies the conditions:

[$N_1$] $|e| = 0 \Leftrightarrow e = 0$

[$N_2$] $|\lambda e| = \lambda |e|$ for all $e \in \Pi[0]$ and $\lambda \geq 0$

[$N_3$] $|e_1 + e_2| \geq |e_1| + |e_2|$ for all $e_1, e_2 \in \Pi[0]$.

The triplet $(W, \Pi, |\cdot|)$ defines a *S.a. normed space*.

If $|\cdot| : \Pi[0] \longrightarrow \mathbb{R}_+$ is a S.a. norm, then $\rho : \Pi \longrightarrow \mathbb{R}_+$ of values

$$\rho(e_1, e_2) = |e_2 - e_1|$$

is a S.a. metric, hence the linear S.a. normed spaces are horistological worlds. These spaces are essential in completions of the classical theory of duality, with applications to concave optimization and $L_p$ duality if $p < 1$ (see [BT4], [B-C], [CB], etc.).

**1.12. Indefinite inner products**. Let $W$ be a linear space over $\Gamma$, which is either $\mathbb{R}$ or $\mathbb{C}$. Function $\langle .,. \rangle : W \times W \longrightarrow \Gamma$ is an *inner product* on $W$ if it satisfies the conditions:

[$I_1$] $\langle \alpha e_1 + \beta e_2, e_3 \rangle = \alpha \langle e_1, e_3 \rangle + \beta \langle e_2, e_3 \rangle$

[$I_2$] $\langle e_1, e_2 \rangle = \overline{\langle e_2, e_1 \rangle}$.

An immediate consequence of [$I_2$] is $\langle e, e \rangle \in \mathbb{R}$, so that either *sign* $\langle e, e \rangle$ is constant, or there exist $e_1, e_2 \in W$ such that $\langle e_1, e_1 \rangle > 0$ and $\langle e_2, e_2 \rangle < 0$. In the first case $\langle .,. \rangle$ is called *scalar product*, or (*semi-) definite inner product*, and it generates the Geometry and the classical Hilbert structure of $W$. In particular, if $\langle e, e \rangle > 0$ for all $e \in W$, then function $\|\cdot\| : W \longrightarrow \mathbb{R}_+$, of values $\|e\| = \sqrt{\langle e, e \rangle}$ is a norm on $W$. Consequently, the forthcoming structure of $W$ is topological.

In the contrary case, when $\langle .,. \rangle$ may change the sign, it is named *indefinite inner product*, and the above construction of a norm is no longer realizable. However, the classical analysis is possible on topological structures that follow from particular features of $(W, \langle .,. \rangle)$ (e.g. decomposability, see [BJ], etc.).

We find a remarkable example of indefinite inner product in the relativist worlds of events $W = \mathbb{R} \times \mathbb{R}^n$, $n = 1,2,3$. The inner product of the events $e_1 = (t_1, s_1)$ and $e_2 = (t_2, s_2)$ has the value

$$\langle e_1, e_2 \rangle = c^2 t_1 t_2 - s_1 s_2.$$

Instead of a Geometry, it generates the entire Chronometry of space-time, giving complete physical interpretations in real variables (see [CJ], [GR], [NG], [C-B], etc.). In particular, if $e = (t, s) \in K[0]$, i.e. $e$ is *time-like* and $t \geq 0$, where $K$ denotes the *causal order* of $W$, then function $|\cdot|_t : K[0] \longrightarrow \mathbb{R}_+$, of values

$$|e|_t = \sqrt{c^2 t^2 - s^2},$$

is a S.a. norm. Consequently, this indefinite inner product naturally leads to horistological structures of space-time; following [B-P1], the same structure appears in arbitrary indefinite inner product spaces. The presence of horistological structures does not exclude topologies on sub-

manifolds of space-time. In particular, if $(t, s) \in K[0]$, then $e^{\perp}$ is a spatial subspace on which the spatial norm
$|e|_s = \sqrt{s^2 - c^2t^2}$ ,
induces an Euclidean topology.

During the last decades, more and more scientists claim that space-time is discrete (e.g. search for "discrete space-time" on the web), and reconstruct the classical theory without underlying continuity. This is another reason to consider that the horistologies, which are so adequate to worlds of events, represent structures of discreteness.

## 2. Discrete functions

The condition of discreteness of a function is dual to continuity, and from the very beginning (see [$BT_2$]) it was conceived to describe the morphisms of the horistological structures. Differently from continuity, which makes sense in topological structures, the discrete functions are defined in a horistological framework. Consequently, continuity and discreteness of a function are not contradictory to each other, but refer to different types of structures. To explicit this relation, we start by recalling the well-known condition of continuity:

**2.1. Continuous functions**. Let $(X, \tau)$ and $(Y, \theta)$ be topological spaces. We say that a function $f : X \longrightarrow Y$ is *continuous* at a point $x \in X$ if
$(\forall)\ U \in \theta\big(f(x)\big)\ (\exists)\ V \in \tau(x)$ such that $([y \in V] \Longrightarrow\ [f(y) \in U])$.

This means that the counter-images trough $f$ of the neighborhoods of $f(x)$ are neighborhoods of $x$, briefl
$f^{\leftarrow}\Big(\theta\big(f(x)\big)\Big) \subseteq \tau(x)$.

The same "scheme of counter-images" defines the uniform continuity, when $X$ and $Y$ are endowed with uniform topologies. The convergence of a net $f$, when $X = D$ is a directed set, works on the same scheme, and reduces to the continuity of the prolongation $\bar{f}$ at $\infty$, where $\overline{D} = D \cup \{\infty\}$ carries an adequate topology. In particular, if $X = \mathbb{N}$, the convergence of a sequence, corresponds to the continuity on the compactification $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

A natural temptation is to apply the same scheme to spaces endowed with horistological structures, i.e. ask the counter-images of the perspectives of $f(e)$ to be perspectives of $e$. So we obtain:

**2.2. h-continuity**. Let $(X, \chi)$ and $(Y, \psi)$ be horistological worlds. We say that a function $f : X \longrightarrow Y$ is *h-continuous* at an event $e \in X$ if
$f^{\leftarrow}\Big(\psi\big(f(e)\big)\Big) \subseteq \chi(e)$.

In spite of several "good properties" of the h-continuous functions, it presents more defects. Following [$BT_2$], the criticism of h-continuity refers to:

a) It is trivially fulfilled if $f(X) \cap K^0(\psi)[f(e)] = \phi$

b) The function of addition in linear spaces is not h-continuous;

c) A relation with the proper orders $K(\chi)$ and $K(\psi)$ holds only for $1{:}\,1$ functions;

d) The constant functions, which are always h-continuous, break down discreteness;

e) The simplest functions in the case $X = Y = \mathbb{R}$ are not h-continuous.

Consequently, we better renounce the continuity-type conditions, and adopt the "scheme of direct images". This one already works for monotony, as well as for boundedness of a function, generally acting between bornologic spaces (see [H-NH], etc.). As a matter of fact, even in a topological framework, we should ask the direct images of discrete sets through discrete functions to be discrete.

**2.3. Definition**. ([$BT_2$]) Let $(W,\chi)$ and $(V,\psi)$ be horistological worlds, and let $f : W \longrightarrow V$ be a function. We say that $l \in V$ is a *germ* of $f$ at $e_0$ (when *e starts from* $e_0$, etc.), and we write $l = germ\, f(e)$ when $e \to e_0$ .
if for every $P \in \chi(e_0)$ we have $f(P) \in \psi(l)$. If $germ\, f(e)$ when $e \to e_0$ exists and
$germ\, f(e) = f(e_0)$ when $e \to e_0$
then we say that $f$ is *discrete* at $e_0$. This means that $(P) \in \psi\big(f(e_0)\big)$ whenever $P \in \chi(e_0)$, briefly
$f\big(\chi(e_0)\big) \subseteq \psi\big(f(e_0)\big)$ .

If $(W,\mathcal{H})$ and $(V,\mathcal{U})$ are uniform horistological worlds, we similarly define the *uniform discreteness* of *f*, namely

$$f_{II}(\mathcal{H}) \stackrel{\text{def}}{=} \{f_{II}(\pi) : \pi \in \mathcal{H}\} \subseteq \mathcal{U} ,$$

where $f_{II}(\pi) = \big\{\big(f(e_1), f(e_2)\big) : (e_1,e_2) \in \pi\big\}$ .

If $X = D$ is directed, and $\overline{D} = D \cup \{\infty\}$ then we define the horistology $\chi$ on $D$ by

$$\chi(a)=\begin{cases}\{\phi\} & if\ a \in D\\ \{P \subset D :\ \exists b \in D\ such\ that\ P \subseteq (\leftarrow, b)\} & if\ a = \infty\end{cases}$$

If the prolongation $\bar{f} : \overline{D} \to Y$ , defined by

$$\bar{f}(a) = \begin{cases} f(a) & if\ a \in D\\ l & if\ a = \infty\end{cases} ,$$

is discrete at $\infty$, then we say that $l$ is a *germ* of *f*, respectively the net $f$ is *emergent* from $l$.

**2.4. General properties of the discrete functions:**

*(i)* The (u.) discrete functions preserve the proper orders, i.e.
$(e_1,e_2) \in K(\chi) \Longrightarrow \big(f(e_1), f(e_2)\big) \in K(\quad)\psi;$
*(ii)* Composing (u.) discrete functions gives a (u.) discrete function;
*(iii)* Function $f : W \to V$ is discrete at $e_0 \in W$ if and only if condition
$[(e_d)_{d\in D}\ emergent\ from\ e_0] \Longrightarrow [f(e_d)_{d\in D}\ emergent\ from\ f(e_0)]$
holds for every net $(e_d)_{d\in D}$ .

A general topic where we deal with discrete functions concerns the comparison of the (u.) horistologies and the operations with (u.) horistological worlds (see [$BT_3$]). The analogy to topological structures is obvious.

**2.5. Category HOR**. In principle, it is very important that the family of all horistological worlds forms a category, called HOR. This fact results from the above Property 2.4. (ii) and the simple remark that the identical function is discrete on each horistological world, i.e. the discrete functions represent the morphisms of this category. Property 2.4. (i) shows that category HOR is a sub-category of ORD.

**2.6. Comparing horistological structures**. Let $\chi$ and $\psi$ be horistologies on the same set $W$. We say that $\chi$ is *finer* than $\psi$ if $\chi(e) \sqsupseteq \psi(e)$ at each $e \in W$, and we note $\chi \sqsupseteq \psi$. In other words, $\chi \sqsupseteq \psi$ holds if and only if the identical function $\iota : (W, \psi) \longrightarrow (W, \chi)$ is discrete on $W$. An immediate consequence of $\chi \sqsupseteq \psi$ is $K(\chi) \supseteq K(\psi)$, and conversely, restricting $\chi$ to smaller order relations leads to a coarser horistology.

By analogy, if $\mathcal{H}$ and $\mathcal{U}$ are uniform horistologies on $W$, we consider that $\mathcal{H}$ is finer than $\mathcal{U}$ if $\mathcal{H} \sqsupseteq \mathcal{U}$. Obviously, this relation holds if and only if the identical function is uniformly discrete.

Let $\rho : \Pi \longrightarrow \mathbb{R}_+$ and $\sigma : K \longrightarrow \mathbb{R}_+$ be S.a. metrics on $W$. If $\Pi \subseteq K$, and

$$\rho(e_1, e_2) \geq \sigma(e_1, e_2)$$

holds for some $k > 0$ at each pair $(e_1, e_2) \in W^2$, then the identical function is u. discrete.

Relation of fineness is an order relation on the set of all (u.) horistologies on $W$. In [BT$_3$] we find a construction of the infimum for arbitrary families of horistological structures.

**2.7. Induced horistological structures**. Let $(W, \mathcal{H})$ be a u. horistological world. If $f : X \to W$ is an arbitrary function, then the *inverse image* of $\mathcal{H}$ through $f$ is the finest u. horistology for which $f$ is u. discrete. On the other hand, if function $g : W \longrightarrow Y$ is injective, then the *direct image* of $\mathcal{H}$ through $g$ is the coarsest u. horistology on $Y$ for which $g$ is u. discrete.

We may extend the construction of the inverse and the direct images to arbitrary families of u. horistological worlds using adequate families of discrete functions.

**2.8. Horistological subspaces, products and quotients**. In practice, we apply the general constructions from above to derive horistologies on subspaces, product and quotient spaces. For instance, if $(W, \mathcal{H})$ is a u. horistological world and $X \subset W$, then the embedding $f : X \to W$ induces a u. horistology on $X$, as an inverse image of $\mathcal{H}$. This u. horistology makes $X$ a *u. horistological subspace* of $W$. This is the finest u. horistology on $X$ that makes the embedding u. discrete.

Similarly, let $I$ be an arbitrary family of indices, and let $X = \prod_{i \in I} W_i$, where $(W_i, \mathcal{H}_i)$ are u. horistological worlds. The finest u. horistology on $X$, for which the projections $f_i : X \longrightarrow \mathcal{H}_i$ are u. discrete is the *product u. horistology*.

The construction of a quotient needs extra precautions. If $(W, \mathcal{H})$ is a u. horistological world and $\theta$ is an equivalence on $W$, we ask $\mathcal{H}$ to be *stable* relative to $\theta$ , which means that $\mathcal{H}$ has a base $\mathcal{B}$ such that $\pi \circ \theta \subseteq \pi$ and $\theta \circ \pi \subseteq \pi$ hold for all $\pi \in \mathcal{B}$. The quotient u. horistology on $W / \theta$ is the *direct image* of $\mathcal{H}$ through the canonical map $x \overset{q}{\to} \hat{x}$. It is the coarsest u. horistology that makes $q$ u. discrete.

Similar constructions are realizable with non-uniform horistologies. In [BT$_3$] we may find a study of the S.a. metrizability of these structures.

In the sequel we discuss the particular case of discrete functions between S.a. metric, S.a. normed, and indefinite inner product spaces, and we put forward several fields of applications.

**2.9. Discrete functions on S.a. metric worlds**. If $(W, \Pi, \rho)$ and $(V, K, \sigma)$ are S.a. metric worlds, then the discreteness of $f : X \to W$ at $e_0 \in W$ takes the $\delta - \varepsilon$ form

$\forall\delta > 0\ \exists\varepsilon > 0$ such that $[\rho(e_0, e) > \delta] \Longrightarrow \big[\rho\big(f(e_0), f(e)\big) > \varepsilon\big]$.

A similar condition defines the uniformly discrete functions. The dilations, i.e. functions for which there exists $k > 0$ such that

$\sigma\big(f(e_1), f(e_2)\big) > k\rho(e_1, e_2)$

holds for all $(e_1, e_2) \in \Pi$ , are examples of u. discrete functions.

If $(W, \Pi, |\cdot|)$ and $(V, K, ]\cdot[)$are S.a. normed linear worlds, and $f : X \to W$ is discrete at 0, then $f$ is u. discrete on $W$. This condition means that

$\forall\delta > 0\ \exists\varepsilon > 0$ such that $[|e| > \delta] \Longrightarrow []f(e)[ > \varepsilon]$.

For example, if $f$ is a dilation, i.e. there exists $k > 0$ such that

$$|f(e)| > k]e[$$

for all $e \in \Pi[0]$, then $f$ is discrete at 0. It is easy to see that the linear discrete functions are always dilations.

Without going into details, we mention that many applications of the discrete functions use the notion of polar of a S.a. norm (studied in [$BT_4$]).

**2.10. Strictly plus operators**. Let $(W, \langle\cdot\rangle)$ be an indefinite inner product space. A linear function $f : W \to W$ is called *plus operator* if it carries the non-negative vectors into non-negative ones. It is well known (see [BJ], etc.) that for such an operator there exists a constant

$$\mu(f) = inf\langle f(x), f(x)\rangle\ \ when\ \langle x, x\rangle = 1$$

such that the inequality

$$\langle f(x), f(x)\rangle \geq \mu(f)\langle x, x\rangle$$

holds for all $x \in W$. If $\mu(f) > 0$, we say that $f$ is a *strictly plus* operator, and in this case it carries positive vectors into positive ones.

In event worlds, which are $\Pi_1$ Pontrjagin spaces, the strictly plus operators represent discrete functions relative to the S.a. norm $]x[_t = \sqrt{\langle x, x\rangle}$ and conversely, each discrete function is a strictly plus operator.

We may easily extend the notion of strictly plus operator to *strictly plus functionals* on S.a. normed spaces $(W, \Pi, ]\cdot[)$. In this case we take $V = \mathbb{R}$, $K$ is the usual order of the reals, and $]y[ = y$. The condition of being strictly plus reduces to the existence of $k > 0$ such that

$$f(x) > k]x[$$

for all $x \in W$, which shows that $f$ is (u.) discrete. A remarkable example consists of $W = \mathcal{L}^p$ with $p < 1$, where $\Pi$ is the usual order of functions, and the S.a. norm is defined by

$$]x[ = \left(\int x^p\right)^{1/p}$$

In [CB] we may see that the spaces of all strictly plus functionals on $\mathcal{L}^p$, which forms the *strictly plus dual*, allow representations by $\mathcal{L}^q$, via the same classical relation

$$\frac{1}{p} + \frac{1}{q} = 1$$

Consequently, in the theory of $\mathcal{L}^p$ duality, continuity and discreteness play similar roles depending on $p \geq 1$ respectively $p < 1$.

**2.11. Zeeman's Theorem**. The connection between discrete transformations and order preserving functions is remarkably stronger in inner product spaces, in the sense that the converse of Property 2.4. (i) is true. The former result was the Zeeman's surprising Theorem that "The causality group of the real 4-dimensional space-time consists of Lorentz transformations, transla- tions and dilations" (see [ZEC]). The causal transformations are $1:1$ mappings $f:\mathbb{R}^4 \longrightarrow \mathbb{R}^4$, such that both $f$ and $f^{-1}$ preserve causality (with no assumption on continuity!). On the other hand, the Lorentz transformations are isometries relative to the S.a. temporal metric, hence they are discrete functions. The translations and dilations are discrete too, hence in other terms, Zeeman's Theorem says that "*The causality group of the real 4-dimensional space-time consists of discrete functions*".

The initial Zeeman's theorem was successively extended to more general indefinite inner product spaces, e.g. in [$BT_5$] it is proved in Krein spaces.

**2.12. Discrete instability**. Let $x : T \longrightarrow \chi$ be the state evolution of a dynamical system. It is well-known that the classical Lyapunov stability of this dynamical system represents the continuity of the "initial state - evolution" function $\Psi: \chi \longrightarrow \chi^T$ . In this framework, the instability of the system equals the discontinuity of , but in practice we frequently meet cases where the discreteness of this function is more adequate, e.g. the reversed mathematical pendulum. This variant of instability is called discrete instability, studied in [B-$P_2$]; discrete Lyapunov functions are introduced and studied in [$PM_2$].

## 3 Discrete sets

The general idea of discreteness of a set figure the isolation of its points. Even for finite sets, the property of isolation depends on the structure given on the whole space; in this survey we will refer to either topological, or horistological structures.

**3.1. Discrete sets in topology**. Let $(S,\tau)$be a topological space, $M \subseteq S$, and $x \in M$. We say that $x$ is an *isolated point* of $M$ (respectively $M$ is *discrete* at $x$, etc.) if there exists a neighborhood $V$ of $x$ such that $M \cap V = \emptyset$. We consider that $M$ itself is *discrete* if all of its points are isolated. The set of all isolated points of $M$ represents the *discrete part* of $M$, and we note it $\delta(M)$, hence $M$ is discrete if and only if $M = \delta(M)$. Function $\delta : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$, which attaches to each subset $M$ its discrete part is known as *operator of discreteness*.

The notion of discreteness is based on the idea of *separation*, i.e. $x$ is isolated in $M$ if and only if it is *separated* from $M \setminus \{x\}$, i.e.

$x \in int[\complement(M \setminus \{x\})] = int[(\complement M) \cup \{x\}]$.

In other terms, there exists a strong relation between the operator of discreteness, and the operator $\gamma : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$, which attaches to each $M$ its *separated companion*, noted $\gamma(M) \stackrel{\text{def}}{=} int(\complement M)$ More exactly,

$x \in \delta(M) \Leftrightarrow x \in \gamma(M \setminus \{x\})$.

On the other hand, the relation between the operators $\gamma$ and *int* allows deriving the essential properties of $\gamma$ from the well-known properties of *int* (see [PG], etc.).

**3.2. Properties of $\boldsymbol{\gamma}$**. For all $M, A, B \subseteq S$ we have:

[sep1] $\gamma(S) = \emptyset$

[sep2] $\gamma(M) \subseteq \complement M$

[sep3] $\gamma(A \cup B) = \gamma(A) \cap \gamma(B)$
[sep4] $\gamma[\complement_{\gamma}(M)] = \gamma(M)$.

Conversely, if an operator $\gamma : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$, independently of any topology, obeys conditions [sep1] - [sep4], then it permits the reconstruction of the topology $\tau_{\gamma}$ on $S$. More exactly,
$\tau_{\gamma}(x) = \{V \in \mathcal{P}(S) : x \in \gamma(\complement V)\}$.
Going back to operator , we may conclude that if we know the discrete part for each subset of S, then we may reconstruct the topology $\tau$.

The topologically discrete sets have remarkable properties, but generally speaking, the discrete sets are very rarely used in topology. As a matter of fact, the continuous sets, which are closed and connected, do appear occasionally too, in spite of the fact that topologies are structures of continuity (!). More than this, there are desirable properties in a general sense of discreteness, which do not hold in topological structures:

**3.3. Deficiencies of the topological discreteness**. The most serious failure is that finite sets are not always discrete. For example, this is the case of $S = \mathbb{R}$, where $M = \{0,1\}$ is not discrete at $x = 0$ relative to the topology $\tau_{right}$ , defined by
$\tau_{right}(x) = \{V \in \mathcal{P}(\mathbb{R}) : \exists \varepsilon > 0 \; such\; that\; V \supseteq (x - \varepsilon, \infty)\}$ .

As a consequence of this situation, the property of discreteness is not always preserved neither by continuous nor by discontinuous functions. For example, if $\tau_E$ denotes the Euclidean topology on $\mathbb{R}$, then the identical function $\iota : (\mathbb{R}, \tau_E) \longrightarrow (\mathbb{R}, \tau_{right})$ is (u.) continuous on $\mathbb{R}$. The set $M = \{0,1\}$ is discrete relative to $\tau_E$, but the image $\iota(M) = M$ is not discrete relative to $\tau_{right}$.

On the other hand, function $sign : (\mathbb{R}, \tau_E) \longrightarrow (\mathbb{R}, \tau_{right})$ is discontinuous at 0, and $sign(M) = M$ too.

**3.4.Discrete sets in horistology**. Following [B-P3], discreteness in horistological worlds depends on horistology and order. More exactly, we consider that $\Lambda$ is an order on the horistological world $(W, \chi)$, such that $\Lambda \subset K(\chi)$, and $M$ is a subset of $W$. We say that an event $e \in M$ is $\Lambda$-*detachable* from $M$ (alternatively, $M$ is $\Lambda$-*discrete* at $e$, etc.) if
$M \cap \Lambda^0(e) \in \chi(e)$.
The set of all $\Lambda$ -detachable points of $M$ is called $\Lambda$ -*discrete part* of $M$, and we note it $\partial_{\Lambda}(M)$. If each point of $M$ is $\Lambda$ -detachable, i.e. $\partial_{\Lambda}(M) = M$, then we consider that $M$ is $\Lambda$ -*discrete*. Function $\partial_{\Lambda}: \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, which extracts the $\Lambda$ -discrete part $\partial_{\Lambda}(M)$ of each subset $M \in \mathcal{P}(W)$ is called operator of $\Lambda$ -*discreteness.*

In the case $\Lambda = \mathrm{K}(\chi)$, we may omit mentioning $\Lambda$, and simply speak of detachability, discreteness, etc. Alternatively, we may interpret the $\Lambda$ -discreteness as discreteness relative to the horistology $/ \Lambda(x)$ , of values

$$\chi/\Lambda(x) = \{P \cap \Lambda[x] : P \in \chi(x)\}$$

Relative to this notion of discreteness we mention:

**3.5. Properties of $\partial_\Lambda$** . Let $\chi : W \longrightarrow \mathcal{P}(\mathcal{P}(W))$ be a horistology and let $\Lambda$ be an order on $W$, such that $\Lambda \subset \mathrm{K}(\chi)$. The operator $\partial_\Lambda$ has the properties:
[$d_0$] $\partial_\Lambda(M) \subset M$ for all $M \in \mathcal{P}(W)$;
[$d_1$] $card\ M \in \mathbb{N} \Longrightarrow \partial_\Lambda(M) = M$
[$d_2$] $L \subseteq M \Longrightarrow L \cap \partial_\Lambda(M) \subseteq \partial_\Lambda(L)$
[$d_3$] $\partial_\Lambda(M) \cap \partial_\Lambda(L) \subseteq \partial_\Lambda(M \cup L)$
[$d_4$] $e \in \partial_\Lambda(M) \Leftrightarrow e \in M \cap \partial_\Lambda(\{e\} \cup \Lambda[M \cap \Lambda^0[e]])$
[$d_5$] For all $e \in W$ and $M \subseteq \Lambda^0[e]$ we have
$e \in \partial_\Lambda(\{e\} \cup M) \Leftrightarrow e \in \partial_\Lambda(\{e\} \cup \Lambda[M])$
[$d_6$] $\Pi \subseteq \Lambda \Longrightarrow \partial_\Lambda(M) \subseteq \partial_\Pi(M)$
[$d_7$] $\partial_\Lambda(\partial_\Lambda(M)) = \partial_\Lambda(M)$
We remark that, according to [d1], the finite sets are always discrete in horistological worlds.
To conclude, we consider that the above parallel presentation of discreteness in topological and horistological structures proves that the latter are more fruitful.

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