DECOMPOSITION OF A TANGENT BUNDLE OF A NORMAL ANTI-INVARIANT SUBMANIFOLDS OF A PARAQUATERNIONIC KÄHLER MANIFOLDS

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Abstract:
We introduce normal anti-invariant submanifolds of a paraquaternionic Kähler manifolds and obtain some basic results on their diferential geometry. Also we show that the tangent bundle of a normal anti-invariant submanifolds of a paraquaternionic Kähler manifolds admits the decomposition in complementary orthogonal distribution.
Key words: *anti-invariant submanifold, Kähler manifolds, tangent bundle.*

1. Introduction

The paraquaternionic Kähler manifolds have been introduced and studied by Garcia-Rio, Matsushita and Vazquez-Lorenzo. We think of a paraquaternionic Kähler manifold as a semi-Riemannian manifold endowed with two local almost product structures and a local almost complex structure satisfying some compatibility conditions.

In the present paper we define the normal anti-invariant submanifolds of a paraquaternionic Kähler manifold and obtain some basic results on their diferential geometry. Also, we show that the tangent bundle of a normal anti-invariant submanifold N of a paraquaternionic Kähler manifold $(M,\ \mathbb{V},\ g)$ admits the decomposition (8) where $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are complementary orthogonal distributions on N.

Throughout the paper all manifolds are smooth and paracompact. If M is a smooth manifold then we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $F(M)$-module of smooth sections of the tangent bundle TM of M. Similar notations will be used for any other manifold or vector bundle. If not stated otherwise, we use indices: $a, b, c, \ldots \in \{1,2,3\}$ and $i, j, k, \ldots \in \{1,2, \ldots, n\}$.

Let M be a manifold endowed with a *paraquaternionic structure* $\mathbb{V}$, that is, $\mathbb{V}$ is a rank-3 subbundle of $End(TM)$ which has a local basis $\{J_1, J_2, J_3\}$ on a coordinate neighbourhood $\mathcal{U} \subset M$ satisfying (see Garcia-Rio-Matsushita-Vazquez-Lorenzo [4])

(a) $J_a^2 = \lambda_a I$, $a \in \{1,2,3\}$,
(b) $J_1 J_2 = -J_2 J_1 = J_3$, (1)
(c) $\lambda_1 = \lambda_2 = -\lambda_3 = 1$.

A semi-Riemannian metric g on M is said to be *adapted* to the paraquaternionic structure $\mathbb{V}$ if it satisfies

$g(X,Y) + \lambda_a g(J_aX, J_aY) = 0 \ \forall\, a \in \{1,2,3\},$ (2)

for any $X, Y \in \Gamma(TM)$, and any local basis $\{J_1,\ J_2, J_3\}$ of $\mathbb{V}$. From relation (1) and relation (2) it follows that

$g(J_aX, Y) + g(X, J_aY) = 0 \ \ \forall\, a \in \{1,2,3\}.$ (3)

Now, suppose $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ is a local basis of $\mathbb{V}$ on $\tilde{\mathcal{U}} \subset M$ and $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$. Then we have

$\tilde{J}_a = \sum_{b=1}^{3} A_{ab} J_b$, (4)

where the 3×3 matrix $[A_{ab}]$ is an element of the pseudo-orthogonal group $SO(2,1)$. From (1) and (2) it follows that M is of dimension $4m$ and g is of neutral signature $(2m, 2m)$.

Next, we denote by $\tilde{\nabla}$ the Levi - Civita connection on (M, g). Then the triple $(M, \mathbb{V}, g)$is called a *paraquaternionic Kähler manifold* if $\mathbb{V}$ is a parallel bundle with respect to $\tilde{\nabla}$. This means that for any local basis $\{J_1,\ J_2, J_3\}$ of $\mathbb{V}$ on $\mathcal{U} \subset M$ there exist the 1-forms p, q, r on $\mathcal{U}$ such that (cf. Garcia-Rio-Matsushita-Vazquez-Lorenzo [4])

$(a)\ \ (\tilde{\nabla}_X J_1)Y = q(X)J_2Y - r(X)J_3Y$,

$(b)\ \ (\tilde{\nabla}_X J_2)Y = -q(X)J_1Y + p(X)J_3Y$, (5)

$(c)\ \ (\tilde{\nabla}_X J_3)Y = -r(X)J_1Y + p(X)J_2Y$,

2. Decomposition the tangent bundle of normal anti-invariant submanifolds

Now, we consider a non-degenerate submanifold N of $(M, \mathbb{V}, g)$ of codimension n. Then we say that N is a *normal anti-invariant submanifold* of $(M, \mathbb{V}, g)$ if the normal bundle $TN^{\perp}$ of N is anti-invariant with respect to any local basis $\{J_1,\ J_2, J_3\}$ of $\mathbb{V}$ on $\mathcal{U}$, that is, we have

$J_a(T_xJ^{\perp}) \subset T_xN\ , \ \ \forall\, a \in \{1,2,3\},\ x \in \mathcal{U}^* = \mathcal{U} \cap N.$ (6)

A large class of normal anti-invariant submanifolds is given in the next proposition.

Proposition 1. Any non-degenerate real hypersurface N of (M, g) is a normal anti-invariant submanifold of $(M, \mathbb{V}, g)$.

Proof. From (3) we deduce that $g(J_aU, U) = 0$, for any $U \in \Gamma(TN^{\perp})$ and $a \in \{1,2,3\}$. Hence $J_aU \in \Gamma(TN)$, which proves (6). ■

Next, we examine the structures that are induced on the tangent bundle of a normal anti-invariant submanifold N of $(M, \mathbb{V}, g)$. First, we put $\mathcal{D}_{ax} = J_a(T_xN^{\perp})$ and note that $\mathcal{D}_{1x},\ \mathcal{D}_{2x}$ and $\mathcal{D}_{3x}$ are mutually orthogonal non-degenerate n - dimensional vector subspaces of T_xN, for any $x \in N$. Indeed, by using (3), $(1b)$ and (6) we obtain

$g(J_1X, J_2Y) = -g(X, J_1J_2Y) = -g(X, J_3Y) = 0,\ \forall\, X, Y \in \Gamma(TN^{\perp})$,

which shows that $\mathcal{D}_{1x}$ and $\mathcal{D}_{2x}$ are orthogonal. By a similar reason we conclude that $\mathcal{D}_{ax}$ and $\mathcal{D}_{bx}$ are orthogonal for any $a \neq b$. Then we can state the following.

Proposition 2. Let N be a normal anti-invariant submanifold of $(M, \mathbb{V}, g)$ of codimension n. Then we have the assertions:

(i) The subspaces $\mathcal{D}_{ax}$ of T_xN satisfy the following

$J_a(\mathcal{D}_{ax}) = T_xN^{\perp}$ and $J_a(\mathcal{D}_{bx}) = \mathcal{D}_{cx}$,

for any $x \in \mathcal{U}^*, a \in \{1,2,3\}$ and any permutation (a, b, c) of $(1, 2, 3)$.

(ii) The mapping

$\mathcal{D}^{\perp}: x \in N \longrightarrow \mathcal{D}_x^{\perp} = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}$,

defines a non-degenerate distribution of rank $3n$ on N.

(iii) The complementary orthogonal distribution $\mathcal{D}$ to $\mathcal{D}^{\perp}$ in TN is invariant with respect to the paraquaternionic structure , that is, we have

$J_a(\mathcal{D}_x) = \mathcal{D}_x \;\; \forall\, x \in \mathcal{U}^*, a \in \{1,2,3\}.$

Proof. First, by using (1) we obtain the assertion (i). Next, by (4) and taking into account that J_a, $a \in \{1,2,3\}$, are automorphisms of $\Gamma(TN)$ and $\mathcal{D}_{ax}$, $a \in \{1,2,3\}$ are mutually orthogonal subspaces we get the assertion (ii). Now, we note that the tangent bundle of M along N has the following orthogonal decompositions:

$$TM = TN \oplus TN^{\perp} = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus TN^{\perp}. \qquad (7)$$

Then we take $Y \in \Gamma(\mathcal{D}^{\perp})$ and by the assertion (i) we deduce that

$$J_aY \in \Gamma(\mathcal{D}^{\perp} \oplus TN^{\perp}),\; \forall\, a \in \{1,2,3\}.$$

On the other hand, if $Y \in \Gamma(TN^{\perp})$, by (6) and the assertion (ii) we infer that

$$J_aY \in \Gamma(\mathcal{D}^{\perp}),\; \forall\, a \in \{1,2,3\}.$$

Thus by using (3) and the second equality in (7) we obtain

$$g(J_aX, Y) = -g(X, J_aY) = 0,\; \forall\, a \in \{1,2,3\},$$

for any $X \in \Gamma(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D}^{\perp} \oplus TN^{\perp})$. Hence $J_aX \in \Gamma(\mathcal{D})$ for any $a \in \{1,2,3\}$ and $X \in \Gamma(\mathcal{D})$, that is, $\mathcal{D}$ is invariant with respect to the paraquaternionic structure $\mathbb{V}$. This completes the proof of the proposition. ■

By assertion (iii) of the above proposition we are entitled to call $\mathcal{D}$ the *paraquaternionic distribution* on N. Also, we note that the paraquaternionic distribution in non-trivial, that is $\mathcal{D} \neq \{0\}$, if and only if $\dim N > 3n$.

Let N be a normal anti-invariant submanifold of codimension n of a $4m$-dimensional paraquaternionic Kähler manifold $(M, \mathbb{V}, g)$. Then according to the definitions of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ we have the orthogonal decomposition

$$TN = \mathcal{D} \oplus \mathcal{D}^{\perp} \qquad (8)$$

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